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Abstract: We give sufficient conditions to establish central limit theorems and moderate deviation principles for support estimates of Poisson point processes. The considered estimates write as linear combinations of extreme values of the point process. Our results are illustrated on four particular cases: Haar and trigonometric series estimates, Faber-Shauder estimate and kernel estimate. A hierarchy between these estimators is proposed by comparing their optimal convergence rates.

Key-words: Functional estimate, Central limit theorem, Moderate deviation principle, Extreme values, Poisson process, Shape estimation.

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Théorèmes limites pour l'estimation à base de valeurs extrêmes du contour de processus ponctuels

Résumé : Nous donnons des conditions suffisantes permettant d'établir un théorème central limite ainsi qu'un principe de déviations modérées pour des estimateurs du support de processus de Poisson. Les estimateurs considérés s'écrivent comme une combinaison linéaire de valeurs extrêmes du processus ponctuel. Nos résultats sont illustrés dans quatre cas : méthode des fonctions orthogonales (base de Haar et base trigonométrique), estimateur utilisant la base de Faber-Shauder et méthode du noyau. Nous proposons une hiérarchie parmi cette famille d'estimateurs basée sur la comparaison des vitesses optimales de convergence.

Mots-clés : estimation fonctionnelle, théorème central limite, principe de déviations modérées, valeurs extrêmes, processus de Poisson, estimation de contour.

1 Introduction

Many proposals are given in the literature for estimating a set S given a finite random set N of points drawn from the interior. This problem of edge or support estimation arises in classification (HARDY & RASSON [20]), clustering problems (HARTIGAN [21]), discriminant analysis (BAUFAYS & RASSON [2]), and outliers detection (DEVROYE & WISE [6]). Applications are also found in image analysis (KOROSTELEV & TSYBAKOV [26]). For instance, the segmentation problem can be considered under the support estimation point of view, where the support is a connex bounded set in $\mathbb{R}^2$. We also point out some applications in econometrics (e.g. DEPRINS, *et al* [5]). In such cases, the unknown support can be written

$$S = \{(x, y) : \quad 0 \leq \ x \leq \ 1 \ \ ; \quad 0 \leq \ y \ \leq \ f(x)\}, \tag{1.1}$$

where f is an unknown function. Here, the problem reduces to estimating f, called the production frontier (see for instance HÄRDLE *et al* [17]). The data consist of pair (X, Y) where X represents the input (labor, energy or capital) used to produce an output Y in a given firm. In such a framework, the value $f(x)$ can be interpreted as the maximum level of output which is attainable for the level of input x. KOROSTELEV *et al* [25] suppose f to be increasing and concave, from economical considerations, which suggests an adapted estimator, called the DEA (Data Envelopment Analysis) estimator. Its asymptotic distribution is established by GIJBELS *et al* [11].

Here N is a Poisson point process, with observed points belonging to a subset S defined as in (1.1) where f is an unknown function which needs not to be monotone. An early paper was written by GEFFROY [9] for independent identically distributed observations from a density ϕ. The proposed estimator is a kind of histogram based on the extreme values of the sample. This work was extended in two main directions.

(a) On the one hand, piecewise polynomials were introduced and their optimality in an asymptotic minimax sense is proved under weak assumptions on the rate of decrease α of the density ϕ towards 0 by KOROSTELEV & TSYBAKOV [26] and by HÄRDLE *et al* [18]. Extreme values methods are then proposed by HALL *et al* [16] and by GIJBELS & PENG [10] to estimate the parameter α.

(b) On the other hand, different propositions for smoothing Geffroy's estimate were made. GIRARD & JACOB [14] introduced estimates based on kernel regressions and orthogonal series method [12, 13]. In the same spirit, GARDES [8] proposed a Faber-Shauder estimate. In each case, the consistency and the limit distribution of the estimator are established.

We also refer to ABBAR [1] and JACOB & SUQUET [23] who used a similar smoothing approach, although their estimates are not based on the extreme values of the Poisson process.

The work presented here offers a general framework for studying the estimates of the family (b). We consider estimates writing as linear combinations of the extreme values of the Poisson process. We establish general central limit theorems and moderate deviation principles for our estimates and we apply them to the particular cases quoted in (b). Such an approach allows to propose a hierarchy within this family of estimates according to their convergence rates.

2 The boundary estimate

Let $S \subset \mathbb{R}^2$ and consider a sequence of Poisson point processes

$$N_n = \{N_n(D) : D \in \mathcal{B}(S)\}, \quad n \geq 1,$$

with intensity measure $nc\lambda$, where $c > 0$, and λ is the Lebesgue measure on $\mathcal{B}(S)$, the Borel σ-algebra of S. Let $\{(X_{n,i}, Y_{n,i}),\ 1 \leq i \leq N_n\}$ be the set of points associated to the point process. In the following, we assume that there exists a function $f : [0,1] \to \mathbb{R}^+$, such that S can be written as (1.1). Our aim is then to estimate S via an estimation of f. Let $k_n \uparrow \infty$ and define by $\{I_{n,r} :\ 1 \leq r \leq k_n\}$ a mesurable equidistant partition of $[0,1]$. For all $1 \leq r \leq k_n$, set

$$D_{n,r} = \{(x,y) : x \in I_{n,r},\ 0 \leq y \leq f(x)\},$$

the cell of S built on $I_{n,r}$ and $N_{n,r} = N_n(D_{n,r})$. We introduce

$$Y^*_{n,r} = \max\{Y_{n,i} : (X_{n,i}, Y_{n,i}) \in D_{n,r}\},$$

if $D_{n,r} \neq \emptyset$ and $Y^*_{n,r} = 0$ otherwise. We use the convention $0 \times \infty = 0$. Our estimator of f is

$$\hat{f}_n : x \in [0,1] \mapsto \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) \left(1 + \frac{1}{N_{n,r}}\right) Y^*_{n,r}, \tag{2.1}$$

where $\kappa_{n,r} : [0,1] \to \mathbb{R}$ is an arbitrary weighting function determining the nature of the estimate. In the next section, conditions are imposed on $\kappa_{n,r}$ and examples are provided in Section 5. It is well-known that $Y^*_{n,r}$ is an estimator of the maximum of f on $I_{n,r}$ with negative bias. The use of the random variable $(1 + N^{-1}_{n,r})Y^*_{n,r}$ allows to reduce this bias. Therefore, $\hat{f}_n$ appears as a linear combination of extreme value estimates of sampled values of f. The asymptotic properties of $\hat{f}_n$ are established in Section 3, and proved in Section 4. Illustrations are presented in Section 5 on the estimates considered in family (b).

3 Main results

Define $m = \min\{f(x) : x \in [0,1]\}$, $M = \max\{f(x) : x \in [0,1]\}$ and

$$\kappa_n(x) = \left(\sum_{r=1}^{k_n} \kappa^2_{n,r}(x)\right)^{1/2}.$$

For all $1 \le r \le k_n$, set $m_{n,r} = \min\{f(x) : x \in I_{n,r}\}$, $M_{n,r} = \max\{f(x) : x \in I_{n,r}\}$ and

$$w_{n,r}(x) = \kappa_{n,r}(x)/\kappa_n(x). \tag{3.1}$$

We consider the following series of assumptions:

(H.1) $k_n \uparrow \infty$ and $k_n/n \to 0$ as $n \to \infty$.
(H.2) $0 < m \le M < +\infty$ and

$$\Delta_n := \max_{1 \le r \le k_n} |M_{n,r} - m_{n,r}| = o\,(k_n/n) \text{ as } n \to \infty.$$

(H.3) For each $(x_1, ..., x_d) \subset [0,1]$, there exists a covariance matrix in $\mathbb{R}^d$

$$\Sigma_{(x_1,\ldots,x_d)} = [\sigma_{i,j}]_{1 \le i,j \le d}$$

such that for all $1 \le i,\ j \le d$,

$$\sum_{r=1}^{k_n} w_{n,r}(x_i) w_{n,r}(x_j) \to \ \sigma_{i,j} \text{ as } n \to \infty.$$

(H.4) For all $x \in [0,1]$,

$$\max_{1 \le r \le k_n} |w_{n,r}(x)| \to 0 \text{ as } n \to \infty.$$

(H.5) For all $x \in [0,1]$,

$$\left| \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) - 1 \right| = o\left(\frac{\kappa_n(x)}{n}\right) \text{ as } n \to \infty.$$

(H.6) For all $x \in [0,1]$,

$$\frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x)(f(x) - m_{n,r}) = o\left(\frac{\kappa_n(x)}{n}\right) \text{ as } n \to \infty.$$

(H.7) For all $x \in [0,1]$,

$$\frac{1}{k_n} \sum_{r=1}^{k_n} |\kappa_{n,r}(x)| \max\left(\Delta_n, \exp\left(-m\frac{nc}{k_n}\right)\right) = o\left(\frac{\kappa_n(x)}{n}\right) \text{ as } n \to \infty.$$

Before proceeding, let us comment the assumptions. **(H.1)**–**(H.4)** are devoted to the control of the centered estimator $(\hat{f}_n - \mathrm{E}\,\hat{f}_n)$. Assumption **(H.1)** imposes that the mean number of points in each $D_{n,r}$ goes to infinity. **(H.2)** controls the variations of the function f on $I_{n,r}$. More precisely it imposes that the mean number of points in $D_{n,r}$ above $m_{n,r}$ converges to 0. (**H.3**) is devoted to the multivariate aspects of the limit theorems. (**H.4**) imposes to all

the weight functions $\kappa_{n,r}(x)$ in the linear combination (2.1) to be approximatively of the same order. This is a natural condition to obtain an asymptotic Gaussian behavior. These assumptions are easy to verify in practice since they involve either f or $\kappa_{n,r}$ without mixing these two quantities. Assumptions **(H.5)**–**(H.7)** are devoted to the control of the bias term $(\mathrm{E}\,\hat{f}_n - f)$. They prevent it to be too important with respect to the variance of the estimate (which will reveal to be of order κ_n/n). Consequently, these three assumptions involve both the unknown function f and the weight functions $\kappa_{n,r}$. **(H.5)** imposes the mean of the coefficients in (2.1) to be approximatively 1. So, the estimate (2.1) can roughly be interpreted as a convex combination of extreme values. **(H.6)** is a localization condition: The weight function $\kappa_{n,r}$ should be concentrated on $I_{n,r}$. Then, the choice of proper weight functions requires to balance the antagonistic conditions **(H.4)** and **(H.6)**. Finally, **(H.7)** can be seen as a stronger version of **(H.2)**.
Our first result gives the multivariate central limit theorem for $(\hat{f}_n)$.

Theorem 1 *Under assumptions* **(H.1)**–**(H.7)**, *and for all* $(x_1, \dots, x_d) \subset [0,1]$,

$$\left\{ \frac{nc}{\kappa_n(x_j)} (\hat{f}_n(x_j) - f(x_j)) : 1 \le j \le d \right\} \xrightarrow[d]{} N(0, \Sigma_{(x_1,\dots,x_d)}).$$

We present now some large deviation properties of $\left(\hat{f}_n\right)$ (see Definition 1 of the appendix for the definition of the large deviation principle and see e.g. Dembo & Zeitouni [4] for a general account on that topic).
For all $(x_1, \dots, x_d) \subset [0,1]$ such that $\Sigma_{(x_1,\dots,x_d)}$ is regular, define

$$\mathrm{I}_{(x_1,\dots,x_d)} : u \in \mathbb{R}^d \mapsto \frac{1}{2}\ {}^t u \Sigma^{-1}_{(x_1,\dots,x_d)} u.$$

The following family of large deviation principles is sometimes referenced in the literature as a moderate deviation principle. It could be used to measure the asymptotic performance of $\hat{f}_n$, see e.g. Kallenberg [24] and references therein.

Theorem 2 *Under assumptions* **(H.1)**–**(H.7)** *and for all* $(x_1, \dots, x_d) \subset [0,1]$ *and each* $\varepsilon_n \downarrow 0$ *such that* $\Sigma_{(x_1,\dots,x_d)}$ *is regular and*

$$\max_{1 \le r \le k_n} \max_{1 \le j \le d} |w_{n,r}(x_j)| = o\left(\sqrt{\varepsilon_n}\right),$$

the sequence of random vectors

$$\left\{ \sqrt{\varepsilon_n} \frac{nc}{\kappa_n(x_j)} (\hat{f}_n(x_j) - f(x_j)) : 1 \le j \le d \right\}$$

follows the large deviation principle in $\mathbb{R}^d$ *with speed* (ε_n) *and good rate function* $I_{(x_1,\dots,x_d)}$.

In practice, c is not known and has to be estimated. In this aim, we introduce

$$\hat{c}_n = \frac{N_n}{n\hat{a}_n},$$

where

$$\hat{a}_n = \frac{1}{k_n}\sum_{r=1}^{k_n}\left(1+\frac{1}{N_{n,r}}\right)Y_{n,r}^*$$

is an estimator of $a = \lambda(S)$. We then have the following corollary:

Corollary 1 *Theorem 1 and Theorem 2 still hold when c is replaced by $\hat{c}_n$.*

4 Proofs

The proofs of our main results are built as follow. First, we establish a multivariate central limit theorem and a moderate deviation principle for the finite dimensional projection of the centered process

$$\frac{nc}{\kappa_n(x)}\left(\hat{f}_n(x) - \mathbb{E}\left(\hat{f}_n(x)\right)\right), \quad x \in [0,1]$$

(see Proposition 1 below). To this aim, by the general framework of the appendix (Theorem 3 and Theorem 4) it is sufficient to control the centered moments of

$$\xi_{n,r} = \left(1+\frac{1}{N_{n,r}}\right)Z_{n,r}^* = \left(1+\frac{1}{N_{n,r}}\right)\left(\frac{nc}{k_n}\right)Y_{n,r}^*.$$

This is achieved in Lemma 2 and Lemma 3. In a second time, we establish that the bias term

$$\frac{nc}{\kappa_n(x)}\left(\mathbb{E}\left(\hat{f}_n(x)\right) - f(x)\right)$$

vanishes when $n \uparrow \infty$ (see Proposition 2). Finally, we prove in Lemma 5 and Lemma 6 that c can be replaced by $\hat{c}_n$ in the multivariate central limit theorem and in the moderate deviation principle. Before that, we introduce some new notations and definitions needed for our proofs. For all $1 \le r \le k_n$, set

$$D_{n,r}^- = \{(x,y) : x \in I_{n,r},\ 0 \le y \le m_{n,r}\},$$

$$D_{n,r}^+ = \{(x,y) : x \in I_{n,r},\ m_{n,r} < y \le f(x)\},$$

$$N_{n,r}^- \text{ (resp. } N_{n,r}^+) = N_n\left(D_{n,r}^-\right) \text{ (resp. } N_n\left(D_{n,r}^+\right)),$$

$$Z_{n,r}^{-} \text{ (resp. } Z_{n,r}^{+}) = \left(\frac{nc}{k_n}\right) \max\left\{Y_{n,i} : (X_{n,i}, Y_{n,i}) \in D_{n,r}^{-} \text{ (resp. } D_{n,r}^{+})\right\},$$

if $D_{n,r}^{-}$ (resp. $D_{n,r}^{+}$) $\neq \emptyset$, and $Z_{n,r}^{-}$ (resp. $Z_{n,r}^{+}$) $= 0$ otherwise. Note that $\xi_{n,r}$ can be expanded as

$$\xi_{n,r} = Z_{n,r}^{-} + \gamma_{n,r}.$$

where

$$\gamma_{n,r} = \gamma_{n,r,1} + \gamma_{n,r,2} + \gamma_{n,r,3},$$

with

$$\gamma_{n,r,1} = \left(Z_{n,r}^{+} - \mathbb{E}\left(Z_{n,r}^{-}\right)\right) \mathbb{I}_{\{N_{n,r}^{+}>0\}},$$

$$\gamma_{n,r,2} = \left(\mathbb{E}\left(Z_{n,r}^{-}\right) - Z_{n,r}^{-}\right) \mathbb{I}_{\{N_{n,r}^{+}>0\}},$$

and

$$\gamma_{n,r,3} = Z_{n,r}^{*}/N_{n,r}.$$

Some technical results are collected in the next lemma.

Lemma 1 *Under assumptions* **(H.1)** *and* **(H.2)** *we have*
i)

$$\max_{1\le r\le k_n} \mathbb{P}\left(N_{n,r}^{+} > 0\right) = O\left(\frac{n}{k_n}\Delta_n\right) = o(1). \tag{4.1}$$

ii) For all $1 \le r \le k_n$, and any $t \in [0, m_{n,r}]$,

$$\mathbb{P}\left(Z_{n,r}^{-} \le t\right) = \exp(t - \lambda_{n,r}), \tag{4.2}$$

with

$$\lambda_{n,r} = \frac{nc}{k_n} m_{n,r}. \tag{4.3}$$

iii) For all $1 \le r \le k_n$,

$$\mathbb{E}\left(Z_{n,r}^{-}\right) = \lambda_{n,r} - \left(1 - e^{-\lambda_{n,r}}\right). \tag{4.4}$$

iv) For all $1 \le r \le k_n$,

$$\mathbb{V}\left(Z_{n,r}^{-}\right) = 1 - 2\lambda_{n,r} e^{-\lambda_{n,r}} - e^{-2\lambda_{n,r}}. \tag{4.5}$$

v) For all $1 \leq r \leq k_n$,

$$\mathbb{E}\left(\frac{Z_{n,r}^-}{N_{n,r}^-}\right) = 1 - e^{-\lambda_{n,r}}\left(1+\lambda_{n,r}\right). \tag{4.6}$$

vi) For all $\ell \geq 1$ *and* $1 \leq r \leq k_n$,

$$\mathbb{E}\left(\left|Z_{n,r}^- - \mathbb{E}\left(Z_{n,r}^-\right)\right|^\ell\right) \leq 1 + \ell!. \tag{4.7}$$

vii) For all $\ell \geq 1$,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r,1}|^\ell\right) \leq \left(1 + \frac{nc}{k_n}\Delta_n\right)^\ell \max_{1\leq r\leq k_n} \mathbb{P}\left(N_{n,r}^+ > 0\right). \tag{4.8}$$

viii) For all $\ell \geq 1$,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r,2}|^\ell\right) \leq (1+\ell!) \max_{1\leq r\leq k_n} \mathbb{P}\left(N_{n,r}^+ > 0\right). \tag{4.9}$$

ix) For all $\ell \geq 1$,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r,3}|^\ell\right) \leq \ell!\left(1 + \left(1+\frac{\Delta_n}{m}\right)^\ell \max_{1\leq r\leq k_n} \mathbb{P}\left(N_{n,r}^+ > 0\right)\right). \tag{4.10}$$

Proof : i), ii), iii) and iv) are obtained by easy calculations.
v) By a well known property of poisson processes, $Z_{n,r}^- \mid N_{n,r}^-$ has the same distribution as the maximum of $N_{n,r}^-$ independent variables uniformly distributed on $[0, \lambda_{n,r}]$. Hence,

$$\begin{aligned}
\mathbb{E}\left(\frac{Z_{n,r}^-}{N_{n,r}^-}\right) &= \mathbb{E}\left(\frac{\mathbb{I}_{\{N_{n,r}^->0\}}}{N_{n,r}^-}\mathbb{E}\left(Z_{n,r}^- \mid N_{n,r}^-\right)\right) = \lambda_{n,r}\mathbb{E}\left(\frac{\mathbb{I}_{\{N_{n,r}^->0\}}}{N_{n,r}^-+1}\right) = \sum_{q=1}^{\infty}\frac{\lambda_{n,r}^{q+1}}{(q+1)!}e^{-\lambda_{n,r}} \\
&= 1 - e^{-\lambda_{n,r}}\left(1+\lambda_{n,r}\right).
\end{aligned}$$

vi) Note that

$$\mathbb{E}\left(\left|Z_{n,r}^- - \mathbb{E}\left(Z_{n,r}^-\right)\right|^\ell\right) = \ell\int_0^{+\infty} t^{\ell-1}\mathbb{P}\left(\left|Z_{n,r}^- - \mathbb{E}\left(Z_{n,r}^-\right)\right| > t\right)dt.$$

Moreover, by (4.2) and (4.4),

$$\begin{aligned}
\mathbb{P}\left(\left|Z_{n,r}^- - \mathbb{E}\left(Z_{n,r}^-\right)\right| > t\right) &= \mathbb{P}\left(Z_{n,r}^- > \mathbb{E}\left(Z_{n,r}^-\right)+t\right) + \mathbb{P}\left(Z_{n,r}^- < \mathbb{E}\left(Z_{n,r}^-\right)-t\right) \\
&= \left[1 - \exp\left(t - \lambda_{n,r} + \mathbb{E}\left(Z_{n,r}^-\right)\right)\right]\mathbb{I}_{[0,\lambda_{n,r}-\mathbb{E}\left(Z_{n,r}^-\right)]}(t) \\
&+ \exp\left(-t - \lambda_{n,r} + \mathbb{E}\left(Z_{n,r}^-\right)\right)\mathbb{I}_{[0,\mathbb{E}\left(Z_{n,r}^-\right)]}(t) \\
&\leq \mathbb{I}_{[0,1]}(t) + \exp\left(e^{-\lambda_{n,r}} - 1\right)e^{-t}.
\end{aligned}$$

Hence,

$$\mathbb{E}\left(\left|Z_{n,r}^{-}-\mathbb{E}\left(Z_{n,r}^{-}\right)\right|^{\ell}\right) \leq \ell \int_{0}^{1} t^{\ell-1} dt+\exp \left(e^{-\lambda_{n,r}}-1\right) \ell \int_{0}^{+\infty} t^{\ell-1} e^{-t} dt$$
$$\leq 1+\exp \left(e^{-\lambda_{n,r}}-1\right) \ell!.$$

vii) It is straightforward from (4.4) that

$$\left|\gamma_{n,r,1}\right|=\left|Z_{n,r}^{+}-\lambda_{n,r}+\lambda_{n,r}-\mathbb{E}\left(Z_{n,r}^{-}\right)\right| \mathbb{I}_{\left\{N_{n,r}^{+}>0\right\}} \leq\left(\frac{nc}{k_{n}} \Delta_{n}+1\right) \mathbb{I}_{\left\{N_{n,r}^{+}>0\right\}}. \tag{4.11}$$

viii) Since $Z_{n,r}^{-}$ and $N_{n,r}^{+}$ are independent,

$$\max_{1 \leq r \leq k_{n}} \mathbb{E}\left(\left|\gamma_{n,r,2}\right|^{\ell}\right) \leq \max_{1 \leq r \leq k_{n}} \mathbb{E}\left(\left|Z_{n,r}^{-}-\mathbb{E}\left(Z_{n,r}^{-}\right)\right|^{\ell}\right) \max_{1 \leq r \leq k_{n}} \mathbb{P}\left(N_{n,r}^{+}>0\right)$$

and the result holds by (4.7).
ix) Since

$$\frac{Z_{n,r}^{*}}{N_{n,r}} \leq \frac{Z_{n,r}^{-}}{N_{n,r}^{-}} \mathbb{I}_{\left\{N_{n,r}^{+}=0\right\}}+\frac{nc}{k_{n}} \frac{M_{n,r}}{N_{n,r}^{-}+1} \mathbb{I}_{\left\{N_{n,r}^{+} \geq 1\right\}},$$

we have,

$$\mathbb{E}\left(\left(\frac{Z_{n,r}^{*}}{N_{n,r}}\right)^{\ell}\right) \leq \mathbb{E}\left(\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)^{\ell}\right)+\mathbb{E}\left(\left(\frac{nc}{k_{n}} \frac{M_{n,r}}{N_{n,r}^{-}+1}\right)^{\ell}\right) \mathbb{P}\left(N_{n,r}^{+} \geq 1\right).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left(\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)^{\ell}\right) &= \mathbb{E}\left(\left(\frac{1}{N_{n,r}^{-}}\right)^{\ell} \mathbb{E}\left(\left(Z_{n,r}^{-}\right)^{\ell} \mid N_{n,r}^{-}\right)\right)=\lambda_{n,r}^{\ell} \mathbb{E}\left(\frac{\mathbb{I}_{\left\{N_{n,r}^{-} \geq 1\right\}}}{\left(N_{n,r}^{-}\right)^{\ell-1}\left(\ell+N_{n,r}^{-}\right)}\right) \\
&= \ell! \sum_{q=1}^{\infty} \frac{(q+\ell-1)!}{\ell! q^{\ell-1} q!} \frac{\lambda_{n,r}^{q+\ell}}{(q+\ell)!} e^{-\lambda_{n,r}} \qquad (4.12)\\
&\leq \ell!,
\end{aligned}$$

since for all ℓ and q,

$$\frac{(q+\ell-1)!}{\ell! q^{\ell-1} q!} \leq 1.$$

Besides,

$$\begin{aligned}
\mathbb{E}\left(\left(\frac{nc}{k_{n}} \frac{M_{n,r}}{N_{n,r}^{-}+1}\right)^{\ell}\right) &= \left(\frac{M_{n,r}}{m_{n,r}}\right)^{\ell} \ell! \sum_{q=0}^{\infty} \frac{(q+\ell)!}{\ell!\,(q+1)^{\ell}\, q!} \frac{\lambda_{n,r}^{q+\ell}}{(q+\ell)!} e^{-\lambda_{n,r}} \leq\left(\frac{M_{n,r}}{m_{n,r}}\right)^{\ell} \ell! \\
&\leq\left(1+\frac{\Delta_{n}}{m}\right)^{\ell} \ell!, \qquad (4.13)
\end{aligned}$$

since for all ℓ and q,

$$\frac{(q+\ell)!}{\ell!\,(q+1)^{\ell}\,q!} \leq 1.$$

Hence,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r,3}|^{\ell}\right) \leq \ell! \left(1+\left(1+\frac{\Delta_n}{m}\right)^{\ell} \max_{1\leq r\leq k_n} \mathbb{P}\left(N^{+}_{n,r}>0\right)\right).$$

■

In the next lemma we give an uniform upper bound on the centered moments of $(\xi_{n,r})$.

Lemma 2 *Under assumptions* **(H.1)** *and* **(H.2)** *we have*

$$\limsup_{n\to\infty} \max_{\ell\geq 2} \frac{1}{(12)^{\ell}\,\ell!} \max_{1\leq r\leq k_n} \mathbb{E}\left(|\xi_{n,r}-\mathbb{E}(\xi_{n,r})|^{\ell}\right) < 1.$$

Proof : By Lemma 7 (see the appendix), we get that for all $1\leq r\leq k_n$,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\xi_{n,r}-\mathbb{E}(\xi_{n,r})|^{\ell}\right) \leq 2^{\ell} \max\left(\max_{1\leq r\leq k_n} \mathbb{E}\left(\left|Z^{-}_{n,r}-\mathbb{E}\left(Z^{-}_{n,r}\right)\right|^{\ell}\right), 2^{\ell} \max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r}|^{\ell}\right)\right). \tag{4.14}$$

From Lemma 1 vi), we have

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(\left|Z^{-}_{n,r}-\mathbb{E}\left(Z^{-}_{n,r}\right)\right|^{\ell}\right) < 2\ell!. \tag{4.15}$$

Moreover, Lemma 7 and Lemma 1 vii)–ix) entail, for all large n, and all $\ell\geq 2$,

$$\max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r}|^{\ell}\right) \leq 3^{\ell} \max_{1\leq j\leq 3} \max_{1\leq r\leq k_n} \mathbb{E}\left(|\gamma_{n,r,j}|^{\ell}\right) \leq 6^{\ell}\ell!,$$

which, combined with (4.14) and (4.15) give the result. ■

The next lemma provides an exact uniform control of the variances of $(\xi_{n,r})_{1\leq r\leq k_n}$.

Lemma 3 *Under assumptions* **(H.1)** *and* **(H.2)** *we have*

$$\max_{1\leq r\leq k_n} |\mathbb{V}(\xi_{n,r})-1| = o(1).$$

Proof : Since

$$\mathbb{V}(\xi_{n,r}) = \mathbb{V}\left(Z^{-}_{n,r}\right) + 2\mathbb{C}ov\left(Z^{-}_{n,r},\gamma_{n,r}\right) + \mathbb{V}(\gamma_{n,r}),$$

we have,

$$\max_{1\le r\le k_n}\left|\mathbb{V}\left(\xi_{n,r}\right)-1\right|\le\max_{1\le r\le k_n}\left|\mathbb{V}\left(Z_{n,r}^{-}\right)-1\right|+\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r}\right)+2\left[\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r}\right)\mathbb{V}\left(Z_{n,r}^{-}\right)\right]^{1/2}.$$

But, by Lemma 1 iv),

$$\max_{1\le r\le k_n}\left|\mathbb{V}\left(Z_{n,r}^{-}\right)-1\right|=\max_{1\le r\le k_n}\left|2\lambda_{n,r}e^{-\lambda_{n,r}}+e^{-2\lambda_{n,r}}\right|=o\left(1\right).$$

Hence, it remains to show that

$$\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r}\right)=o\left(1\right),$$

and even, by Lemma 7, that

$$\max_{1\le j\le 3}\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r,j}\right)=o\left(1\right).$$

Now, Lemma 1 vii) and viii) with $\ell=2$ clearly imply that

$$\max_{1\le j\le 2}\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r,j}\right)=o\left(1\right),$$

so it remains to prove that

$$\max_{1\le r\le k_n}\mathbb{V}\left(\gamma_{n,r,3}\right)=o\left(1\right).$$

By (4.12) with $\ell=2$,

$$\begin{aligned}\mathbb{E}\left(\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)^{2}\right)&=\sum_{q=1}^{\infty}\left(1+\frac{1}{q}\right)\frac{\lambda_{n,r}^{q+2}}{(q+2)!}e^{-\lambda_{n,r}}\\&=1-e^{-\lambda_{n,r}}\left(1+\lambda_{n,r}+\frac{\lambda_{n,r}^{2}}{2}\right)+\sum_{q=1}^{\infty}\frac{1}{q}\frac{\lambda_{n,r}^{q+2}}{(q+2)!}e^{-\lambda_{n,r}}.\end{aligned}$$

Now, since,

$$\max_{1\le r\le k_n}e^{-\lambda_{n,r}}\left(1+\lambda_{n,r}+\frac{\lambda_{n,r}^{2}}{2}\right)\le\max_{\lambda\ge mnc/k_n}e^{-\lambda}\left(1+\lambda+\frac{\lambda^{2}}{2}\right)=o\left(1\right),$$

and for all $Q\ge 1$,

$$\begin{aligned}\max_{1\le r\le k_n}\sum_{q=1}^{\infty}\frac{1}{q}\frac{\lambda_{n,r}^{q+2}}{(q+2)!}e^{-\lambda_{n,r}}&\le\max_{1\le r\le k_n}\sum_{q=1}^{Q}\frac{\lambda_{n,r}^{q+2}}{(q+2)!}e^{-\lambda_{n,r}}+\frac{1}{Q}\max_{1\le r\le k_n}\sum_{q=1}^{\infty}\frac{\lambda_{n,r}^{q+2}}{(q+2)!}e^{-\lambda_{n,r}}\\&\le o\left(1\right)+\frac{1}{Q},\end{aligned}$$

we get,

$$\max_{1\le r\le k_n}\left|\mathbb{E}\left(\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)^2\right)-1\right|=o(1).$$

Furthermore, by Lemma 1 v),

$$\max_{1\le r\le k_n}\left|\mathbb{E}^2\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)-1\right|=\max_{1\le r\le k_n}\left[1-\left(1-e^{-\lambda_{n,r}}\left(1+\lambda_{n,r}\right)\right)^2\right]=o\left(1\right).$$

Therefore,

$$\max_{1\le r\le k_n}\mathbb{V}(\gamma_{n,r,3})\le\max_{1\le r\le k_n}\left|\mathbb{E}\left(\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)^2\right)-1\right|+\max_{1\le r\le k_n}\left|\mathbb{E}^2\left(\frac{Z_{n,r}^{-}}{N_{n,r}^{-}}\right)-1\right|=o\left(1\right).$$

■

Proposition 1 *Under assumptions* **(H.1)**–**(H.4)**,
i) for all $(x_1,...,x_d)\subset[0,1]$,

$$\left\{\frac{nc}{\kappa_n\left(x_j\right)}\left(\hat{f}_n\left(x_j\right)-\mathbb{E}\left(\hat{f}_n\left(x_j\right)\right)\right):1\le j\le d\right\}\underset{d}{\to}N\left(0,\Sigma_{(x_1,\dots,x_d)}\right),$$

ii) for all $(x_1,...,x_d)\subset[0,1]$, *and all* $\varepsilon_n\downarrow 0$ *such that such that* $\Sigma_{(x_1,\dots,x_d)}$ *is regular and*

$$\max_{1\le r\le k_n}\max_{1\le j\le d}\left|w_{n,r}\left(x_j\right)\right|=o\left(\sqrt{\varepsilon_n}\right),$$

$$\left\{\sqrt{\varepsilon_n}\frac{nc}{\kappa_n\left(x_j\right)}\left(\hat{f}_n\left(x_j\right)-\mathbb{E}\left(\hat{f}_n\left(x_j\right)\right)\right):1\le j\le d\right\},$$

follows the large deviation principle in $\mathbb{R}^d$ *with speed* (ε_n) *and good rate function* $I_{(x_1,\dots,x_d)}$.

Proof : For all $1\le r\le k_n$, set

$$\zeta_{n,r}=\xi_{n,r}-\mathbb{E}\left(\xi_{n,r}\right)$$

and

$$w_{n,r}={}^t\left(w_{n,r}\left(x_1\right),...,w_{n,r}\left(x_d\right)\right).$$

Then, it is easily seen that

$$\left\{\frac{nc}{\kappa_n\left(x_j\right)}\left(\hat{f}_n\left(x_j\right)-\mathbb{E}\left(\hat{f}_n\left(x_j\right)\right)\right):1\le j\le d\right\}=\sum_{r=1}^{k_n}w_{n,r}\zeta_{n,r}.$$

Moreover, note that $(\zeta_{n,r})_{1\le r\le k_n}$ are independent. Thus, by **(H.3)**, **(H.4)**, Lemma 3 and Lemma 2 we may apply Theorem 3 and Theorem 4 to get the intended results. ■

In order to control the bias term, we need information on the expectations of $(\xi_{n,r})_{1\le r\le k_n}$.

Lemma 4 *Under assumptions* **(H.1)** *and* **(H.2)** *we have*

$$\max_{1\le r\le k_n} |\mathbb{E}(\xi_{n,r}) - \lambda_{n,r}| = O\left(\frac{n}{k_n}\max\left(\Delta_n, \exp\left(-m\frac{nc}{k_n}\right)\right)\right).$$

Proof : Lemma 1 iii) yields

$$\mathbb{E}(\xi_{n,r}) - \lambda_{n,r} = e^{-\lambda_{n,r}} - 1 + \mathbb{E}(\gamma_{n,r,1}) + \mathbb{E}(\gamma_{n,r,3}), \tag{4.16}$$

with

$$\begin{aligned}\mathbb{E}(\gamma_{n,r,1}) &= \mathbb{E}\left((Z^+_{n,r} - \lambda_{n,r})\mathbb{I}_{\{N^+_{n,r}>0\}}\right) + \mathbb{E}\left((\lambda_{n,r} - Z^-_{n,r})\mathbb{I}_{\{N^+_{n,r}>0\}}\right)\\ &= \mathbb{E}\left((Z^+_{n,r} - \lambda_{n,r})\mathbb{I}_{\{N^+_{n,r}>0\}}\right) + (1-e^{-\lambda_{n,r}})\mathbb{P}(N^+_{n,r}>0),\end{aligned} \tag{4.17}$$

and, in view of Lemma 1 v),

$$\begin{aligned}\mathbb{E}(\gamma_{n,r,3}) &= \mathbb{E}\left(\frac{Z^-_{n,r}}{N^-_{n,r}}\right)\mathbb{P}(N^+_{n,r}=0) + \mathbb{E}\left(\frac{Z^+_{n,r}}{N_{n,r}}\right)\mathbb{P}(N^+_{n,r}>0)\\ &= (1-e^{-\lambda_{n,r}}(1+\lambda_{n,r}))\mathbb{P}(N^+_{n,r}=0) + \mathbb{E}\left(\frac{Z^+_{n,r}}{N_{n,r}}\right)\mathbb{P}(N^+_{n,r}>0).\end{aligned} \tag{4.18}$$

From (4.16)–(4.18), it follows that

$$\mathbb{E}(\xi_{n,r}) - \lambda_{n,r} = \mathbb{E}\left((Z^+_{n,r} - \lambda_{n,r})\mathbb{I}_{\{N^+_{n,r}>0\}}\right) + \mathbb{E}\left(\frac{Z^+_{n,r}}{N_{n,r}}\right)\mathbb{P}(N^+_{n,r}>0) - \lambda_{n,r}e^{-\lambda_{n,r}}.$$

Now, by Lemma 1 i) and (4.13) with $\ell = 1$, we obtain

$$\begin{aligned}\max_{1\le r\le k_n} |\mathbb{E}(\xi_{n,r}) - \lambda_{n,r}| &\le \frac{nc}{k_n}\Delta_n + \left(1+\frac{\Delta_n}{m}\right)\max_{1\le r\le k_n}\mathbb{P}(N^+_{n,r}>0) + \frac{Mnc}{k_n}\exp\left(-m\frac{nc}{k_n}\right)\\ &= O\left(\frac{n}{k_n}\max\left(\Delta_n, \exp\left(-m\frac{nc}{k_n}\right)\right)\right),\end{aligned}$$

under **(H.2)**. ■

Proposition 2 *Under assumptions* **(H.1)**, **(H.2)**, **(H.5)**–**(H.7)**, *we have for all* $x\in[0,1]$,

$$\frac{nc}{\kappa_n(x)}\left(\mathbb{E}\left(\hat f_n(x)\right) - f(x)\right) \to 0.$$

Proof : For all $x \in [0,1]$, we get, by the triangle inequality and assumptions **(H.5)** and **(H.6)**,

$$\begin{aligned}
\frac{nc}{\kappa_n(x)} \left| \mathbb{E}\left(\hat{f}_n(x) \right) - f(x) \right| &= \left| \sum_{r=1}^{k_n} w_{n,r}(x)\, \mathbb{E}(\xi_{n,r}) - \frac{nc}{\kappa_n(x)} f(x) \right| \\
&\leq \left| \sum_{r=1}^{k_n} w_{n,r}(x) \left(\mathbb{E}(\xi_{n,r}) - \lambda_{n,r} \right) \right| \\
&+ \left| \frac{nc}{k_n} \sum_{r=1}^{k_n} w_{n,r}(x) \left(m_{n,r} - f(x) \right) \right| \\
&+ \left| \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) - 1 \right| \frac{ncf(x)}{\kappa_n(x)} \\
&\leq \left(\sum_{r=1}^{k_n} |w_{n,r}(x)| \right) \max_{1 \leq r \leq k_n} |\mathbb{E}(\xi_{n,r}) - \lambda_{n,r}| + o(1). \qquad (4.19)
\end{aligned}$$

Lemma 4 and condition **(H.7)** give the result. ■

Theorem 1 and Theorem 2 are straightforward consequences of Proposition 1 and Proposition 2. The two following lemmas are dedicated to the replacement of c by $\hat{c}_n$ in both theorems.

Lemma 5 *Under assumptions* **(H.1)** *and* **(H.2)**, *for all* $\delta > 0$, *there exist* $\alpha_\delta > 0$ *and* $n_\delta > 0$ *such that*

$$\forall n \geq n_\delta,\ \mathbb{P}(|\hat{c}_n - c| \geq \delta) \leq 3\exp(-n\alpha_\delta).$$

Proof : We have

$$|\hat{c}_n - c| \leq \frac{1}{n\hat{a}_n} |N_n - nac| + ac \left| \frac{1}{\hat{a}_n} - \frac{1}{a} \right|.$$

Let $\delta > 0$ and $\eta_\delta = \min(a/2,\ a\delta/(4c))$. Then,

$$|\hat{c}_n - c| \leq |\hat{c}_n - c| \mathbb{I}_{\{|\hat{a}_n - a| > \eta_\delta\}} + \left(\frac{2}{na} |N_n - nac| + \frac{2c\eta_\delta}{a} \right) \mathbb{I}_{\{|\hat{a}_n - a| \leq \eta_\delta\}},$$

and therefore

$$\begin{aligned}
\mathbb{P}(|\hat{c}_n - c| \geq \delta) &\leq \mathbb{P}\left(\frac{1}{n} |N_n - nac| \geq \frac{a}{2} \left(\delta - \frac{2c\eta_\delta}{a} \right) \right) + \mathbb{P}(|\hat{a}_n - a| > \eta_\delta) \\
&\leq \mathbb{P}\left(\frac{N_n}{na} \notin \left] c - \frac{\delta}{4},\ c + \frac{\delta}{4} \right[\right) + \mathbb{P}(|\hat{a}_n - a| > \eta_\delta). \qquad (4.20)
\end{aligned}$$

Let us consider the first term of (4.20). Since N_n has a Poisson distribution, $N_n \rightsquigarrow \mathcal{P}(nac)$, it can be expanded as $N_n = \sum_{k=1}^{n} \pi_k$, where the random variables π_k are i.i.d. $\mathcal{P}(ac)$. Introducing

$$\Lambda_\pi(s) = \log \mathbb{E}(e^{s\pi}) = ac(e^s - 1)$$

and denoting

$$\Lambda_\pi^*(t) = \sup_{t\in\mathbb{R}}(st - \Lambda_\pi(s)) = \left| \begin{array}{lll} t\log(t/ac) - t + ac & \text{if} & t > 0 \\ +\infty & \text{if} & t \le 0, \end{array} \right.$$

Cramer's theorem (see DEMBO & ZEITOUNI [4], Theorem 2.2.3) yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{N_n}{na} \notin \left]c - \frac{\delta}{4},\ c + \frac{\delta}{4}\right[\right) \le -\inf\left\{\Lambda_\pi^*(t),\ t \notin \left]c - \frac{\delta}{4},\ c + \frac{\delta}{4}\right[\right\} < 0.$$

Consequently, there exists $\alpha'_\delta > 0$ such that

$$\forall n \ge 1, \mathbb{P}\left(\frac{N_n}{na} \notin \left]c - \frac{\delta}{4},\ c + \frac{\delta}{4}\right[\right) \le \exp(-n\alpha'_\delta). \tag{4.21}$$

Consider now the second term of (4.20). Lemma 4 yields

$$|\mathbb{E}(\hat{a}_n) - a| = O\left(\max\left(\Delta_n, \exp\left(-m\frac{nc}{k_n}\right)\right)\right)$$

which converges to 0 under **(H.1)** and **(H.2)**. Therefore, there exists $n_\delta > 0$ such that

$$\forall n \ge n_\delta,\ \mathbb{P}(|\hat{a}_n - a| \ge \eta_\delta) \le \mathbb{P}(|\hat{a}_n - \mathbb{E}(\hat{a}_n)| \ge \eta_\delta/2) = \mathbb{P}\left(\left|\sum_{r=1}^{k_n}(\xi_{n,r} - \mathbb{E}(\xi_{n,r}))\right| \ge \eta_\delta nc/2\right),$$

and in view of Lemma 2, applying Berstein's inequality (see BOSQ [3], Theorem 2.6) yields

$$\forall n \ge n_\delta,\ \mathbb{P}(|\hat{a}_n - a| \ge \eta_\delta) \le 2\exp\left(-\frac{\eta_\delta^2 c^2 n^2}{2304 k_n + 48\eta_\delta cn}\right) \le 2\exp\left(-\frac{\eta_\delta^2 c^2}{2304 + 48\eta_\delta c} n\right). \tag{4.22}$$

Defining $\alpha_\delta = \min(\alpha'_\delta, \eta_\delta^2 c^2/(2304 + 48\eta_\delta c))$ and collecting (4.20)–(4.22) give the result. ■

The last lemma proves that the difference

$$D_n(x) = \frac{n}{\kappa_n(x)}(\hat{c}_n - c)(\hat{f}_n(x) - f(x))$$

can be neglected both in the central limit theorem and in the moderate deviation principle.

Lemma 6 *Let* $(x_1, \dots, x_d) \subset [0, 1]$.
i) Under assumptions of Theorem 1,

$$\{D_n(x_j),\ 1 \le j \le d\} \xrightarrow{\mathbb{P}} 0.$$

ii) Under assumptions of Theorem 2, for all $\eta > 0$,

$$\limsup_{n\to\infty} \varepsilon_n \log \mathbb{P}\left(\sqrt{\varepsilon_n} \max_{1\le j\le d} |D_n(x_j)| \ge \eta\right) = -\infty.$$

Proof : i) For all $\eta > 0$ and $\delta > 0$, we have

$$\mathbb{P}\left(\max_{1\le j\le d}|D_n(x_j)|\ge\eta\right)\le\mathbb{P}(|\hat{c}_n-c|\ge\delta)+\mathbb{P}\left(\max_{1\le j\le d}\frac{nc}{\kappa_n(x_j)}|\hat{f}_n(x_j)-f(x_j)|\ge\eta c/\delta\right),$$

where the first term converges to 0 as $n\to\infty$ in view of Lemma 5. Thus,

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}\left(\max_{1\le j\le d}|D_n(x_j)|\ge\eta\right) &\le \limsup_{n\to\infty}\mathbb{P}\left(\max_{1\le j\le d}\frac{nc}{\kappa_n(x_j)}|\hat{f}_n(x_j)-f(x_j)|\ge\eta c/\delta\right)\\ &= \mathbb{P}\left(\max_{1\le j\le d}|G_j|\ge\eta c/\delta\right), \qquad (4.23)\end{aligned}$$

where $(G_1,\dots,G_d)$ follows the distribution $N(0,\Sigma_{(x_1,\dots,x_d)})$ (see Theorem 1). Letting $\delta\to 0$ in (4.23) concludes the proof:

$$\limsup_{n\to\infty}\mathbb{P}\left(\max_{1\le j\le d}|D_n(x_j)|\ge\eta\right)=0.$$

ii) For all $\eta>0$ and $\delta>0$, we have the well-known inequality (see DEMBO & ZEITOUNI [4], Lemma 1.2.15)

$$\limsup_{n\to\infty}\varepsilon_n\log\mathbb{P}\left(\sqrt{\varepsilon_n}\max_{1\le j\le d}|D_n(x_j)|\ge\eta\right)\le$$

$$\max\left\{\limsup_{n\to\infty}\varepsilon_n\log\mathbb{P}\left(\sqrt{\varepsilon_n}\max_{1\le j\le d}\frac{nc}{\kappa_n(x_j)}|\hat{f}_n(x_j)-f(x_j)|\ge\eta c/\delta\right),\limsup_{n\to\infty}\varepsilon_n\log\mathbb{P}(|\hat{c}_n-c|\ge\delta)\right\}.$$

First, from Theorem 2,

$$\limsup_{n\to\infty}\varepsilon_n\log\mathbb{P}\left(\sqrt{\varepsilon_n}\max_{1\le j\le d}\frac{nc}{\kappa_n(x_j)}|\hat{f}_n(x_j)-f(x_j)|\ge\eta c/\delta\right)\le-\inf_{\|u\|_{\mathbb{R}^d}\ge\eta c/\delta}I_{(x_1,\dots,x_d)}(u)\to-\infty$$

when $\delta\to 0$. Second, Lemma 5 yields for $n\ge n_\delta$,

$$\varepsilon_n\log\mathbb{P}(|\hat{c}_n-c|\ge\delta)\le-\alpha_\delta n\varepsilon_n+\varepsilon_n\log(3)\to-\infty$$

as $n\to\infty$, since for n large enough,

$$n\varepsilon_n\ge n\max_{1\le r\le k_n}w_{n,r}^2(x)\ge n/k_n\to\infty.$$

As a conclusion,

$$\limsup_{n\to\infty}\varepsilon_n\log\mathbb{P}\left(\sqrt{\varepsilon_n}\max_{1\le j\le d}|D_n(x_j)|\ge\eta\right)=-\infty.$$

■

5 Applications

We provide four different illustrations of the general results established in the previous section. We examine a kernel estimator, a particular case of wavelet estimator (the Haar estimate), an orthogonal series method (the trigonometric series method) as well as a non-orthogonal series method (the Faber-Shauder series method). First, the definition and the basic properties of each estimate are recalled. Second, we rewrite the sufficient conditions **(H.1)**–**(H.7)** in a more explicit way in these particular cases. Finally, we propose an illustration of the behavior of each estimate on simulated data.

5.1 Four boundary estimates

In the sequel (b_n) is a sequence of positive real numbers tending to infinity and $x_{n,r}$ is the center of the interval $I_{n,r}$, $r = 1, \ldots, k_n$. The unknown function f is supposed to be twice continuously differentiable.

5.1.1 Kernel estimate

The kernel estimate is based on a nonparametric regression on the set of the bias corrected extreme values $(1 + N_{n,r}^{-1})Y_{n,r}^*$, $r = 1, \ldots, k_n$. For $x \in [0,1]$, it writes

$$\hat{f}_n(x) = \frac{1}{h_n k_n} \sum_{r=1}^{k_n} K\left(\frac{x - x_{n,r}}{h_n}\right)\left(1 + \frac{1}{N_{n,r}}\right) Y_{n,r}^*, \tag{5.1}$$

where K is a Parzen-Rosenblatt kernel, supposed to be bounded, positive, even, twice continuously differentiable and such that $x \to x^2 K(x)$ is integrable. (h_n) is a sequence of positive real numbers tending to zero called the window. It tunes the smoothing introduced by the kernel. For a review on nonparametric regression, see HÄRDLE [19]. It is apparent that (5.1) is a particular case of (2.1) in which $\kappa_{n,r}(x) = b_n K(b_n(x - x_{n,r}))$, with $b_n = 1/h_n$. The estimate $\hat{f}_n$ is at least twice differentiable since it inherits from the regularity of the kernel K.

5.1.2 Haar estimate

Introduce a dyadic subdivision $\{J_\ell,\ \ell = b_n + 1, \ldots, 2b_n + 1\}$ of $[0,1]$ defined by $J_\ell = [c_\ell, d_\ell]$, where $c_\ell = p_\ell/2^{q_\ell - 1}$, $d_\ell = (p_\ell + 1)/2^{q_\ell - 1}$, p_ℓ and q_ℓ are the integers uniquely determined by $\ell = 2^{q_\ell - 1} + p_\ell$ and $0 \le p_\ell < 2^{q_\ell - 1}$. The Haar basis (HAAR [15]) is the orthogonal basis defined by:

$$e_0 = \mathbb{I}_{[0,1]}, \qquad e_\ell = 2^{\frac{q_\ell - 1}{2}}\left(\mathbb{I}_{J_{2\ell}} - \mathbb{I}_{J_{2\ell+1}}\right), \quad \ell \ge 1.$$

We note $S_n(f)$ the expansion of f on the Haar basis truncated to the $(b_n + 1)$ first terms

$$S_n(f)(x) = \sum_{\ell=0}^{b_n} a_\ell e_\ell(x), \qquad x \in [0,1].$$

Each coefficient a_ℓ, $\ell = 0, \ldots, b_n$ is estimated by the random Riemann sum

$$\hat{a}_{\ell,k_n} = \sum_{r=1}^{k_n} e_\ell(x_{n,r}) \left(1 + \frac{1}{N_{n,r}}\right) \frac{Y^*_{n,r}}{k_n},$$

leading to the estimator of f:

$$\hat{f}_n(x) = \sum_{\ell=0}^{b_n} \hat{a}_{\ell,k_n} e_\ell(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} K_n^{\mathrm{H}}(x_{n,r}, x) \left(1 + \frac{1}{N_{n,r}}\right) Y^*_{n,r}, \tag{5.2}$$

where K_n^{H} is the Dirichlet's kernel associated to the Haar basis:

$$K_n^{\mathrm{H}}(x, y) = \sum_{\ell=0}^{b_n} e_\ell(x) e_\ell(y),\ (x, y) \in [0, 1]^2.$$

It appears that (5.2) is a particular case of (2.1) with $\kappa_{n,r}(x) = K_n^{\mathrm{H}}(x_{n,r}, x)$. Let us note that, since the functions of the Haar basis are not continuous, the estimator is of f is also not continuous in general.

5.1.3 Trigonometric estimate

Formally, the trigonometric estimate writes as the Haar estimate:

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} K_n^{\mathrm{T}}(x_{n,r}, x) \left(1 + \frac{1}{N_{n,r}}\right) Y^*_{n,r}, \tag{5.3}$$

except that K_n^{T} is the Dirichlet's kernel associated to the trigonometric basis which is defined by

$$e_0(x) = 1, \qquad e_{2k-1}(x) = \sqrt{2} \cos(2k\pi x), \qquad e_{2k}(x) = \sqrt{2} \sin(2k\pi x), \quad k \geq 1.$$

As previously, we have $\kappa_{n,r}(x) = K_n^{\mathrm{T}}(x_{n,r}, x)$. In this case, the estimator $\hat{f}_n$ is C^∞.

5.1.4 Faber-Shauder estimate

Consider the dyadic subdivision introduced in Subsection 5.1.2. The Faber-Shauder is the non-orthogonal basis of continuous functions defined by

$$e_{-1}(x) = \mathbb{I}_{[0,1]}(x),\ e_0(x) = x\mathbb{I}_{[0,1]}(x),\ e_\ell(x) = 2^{q_\ell}((x - c_\ell)\mathbb{I}_{J_{2\ell}} + (d_\ell - x)\mathbb{I}_{J_{2\ell+1}}),\ \ell \geq 1.$$

Note $S_n(f)$ the expansion of f on the Faber-Shauder basis truncated to the $(b_n + 2)$ first terms

$$S_n(f)(x) = \sum_{\ell=-1}^{b_n} a_\ell e_\ell(x), \qquad x \in [0, 1].$$

Each coefficient a_ℓ, $\ell = -1, \dots, k_n$ is estimated by

$$\hat{a}_{\ell,k_n} = (b_n + 1) \sum_{r=1}^{k_n} g_\ell(x_{n,r}) \left(1 + \frac{1}{N_{n,r}}\right) \frac{Y^*_{n,r}}{k_n},$$

where g_ℓ is a piecewise linear function on $[0,1]$ that we do not precise here. Finally, the estimator can be written as:

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} K_n^{\mathrm{F}}(x_{n,r}, x) \left(1 + \frac{1}{N_{n,r}}\right) Y^*_{n,r}, \tag{5.4}$$

where K_n^{F} is the kernel defined by:

$$K_n^{\mathrm{F}}(x,y) = (b_n + 1) \sum_{\ell=-1}^{b_n} g_\ell(x) e_\ell(y),\ (x,y) \in [0,1]^2.$$

It appears that (5.4) is a particular case of (2.1) with $\kappa_{n,r}(x) = K_n^{\mathrm{F}}(x_{n,r}, x)$. $\hat{f}_n$ is a continuous estimate of f.

5.2 Basic properties

In this paragraph, we provide bounds on the three following quantities:

$$\max_{1\le r\le k_n} |\kappa_{n,r}(x)|,\ \frac{1}{k_n}\sum_{r=1}^{k_n} \kappa_{n,r}(x),\ \kappa_n^2(x) = \sum_{r=1}^{k_n} \kappa_{n,r}^2(x),\ x \in [0,1]$$

that will reveal useful for rewritting the conditions **(H.1)**–**(H.7)**. The results which are collected here are classical ones so we omit their proofs.

5.2.1 Kernel estimate

It is clear that,

$$\max_{1\le r\le k_n} |\kappa_{n,r}(x)| = O(b_n),\ \forall x \in [0,1].$$

Besides, under the condition $b_n^{3/2} = o(k_n)$, Proposition 1 in Girard & Jacob [14] yields

$$\left| \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) - 1 \right| = O\left(1/b_n^2\right) + O\left(b_n^3/k_n^2\right),\ \forall x \in]0,1[,$$

and

$$\kappa_n^2(x) \sim k_n b_n \int_0^1 K^2(t)dt,\ \forall x \in]0,1[. \tag{5.5}$$

5.2.2 Haar estimate

We introduce the following assumptions on the sequences (b_n) and (k_n):

$$b_n + 1 = 2^{b'_n}, \quad b'_n \in \mathbb{N}, \quad k_n = d_n(b_n + 1), \quad d_n \in \mathbb{N}^*. \tag{5.6}$$

The first part of (5.6) is classical. It imposes that the number of terms in the expansion $S_n(x)$ is a power of two. The consequence of the second part of the condition is that for each $\ell \in \{b_n + 1, \dots, 2b_n + 1\}$, J_ℓ is exactly the union of d_n subintervals $I_{n,r}$. Now, let $x \in [0,1]$ and $\ell(x)$ such that $x \in J_{\ell(x)}$. Under (5.6), the Dirichlet's kernel reduces to

$$K_n^{\mathrm{H}}(x, x_{n,r}) = (b_n + 1)\mathbb{I}_{\{x_{n,r} \in J_{\ell(x)}\}}. \tag{5.7}$$

We thus obtain, for all $x \in [0,1]$:

$$\begin{aligned} \max_{1 \le r \le k_n} |\kappa_{n,r}(x)| &= 1 + b_n, \\ \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) &= 1, \\ \kappa_n^2(x) &= k_n(b_n + 1). \end{aligned} \tag{5.8}$$

5.2.3 Trigonometric estimate

Suppose for convenience that b_n is even. Then, the Dirichlet's kernel can be explicitely computed:

$$K_n^{\mathrm{T}}(x,y) = \left| \begin{array}{ll} \dfrac{\sin[(1+b_n)\pi(x-y)]}{\sin[\pi(x-y)]} & x \ne y, \\ 1 + b_n. & x = y, \end{array} \right.$$

and it follows that, for all $x \in [0,1]$,

$$\max_{1 \le r \le k_n} |\kappa_{n,r}(x)| = 1 + b_n,$$

$$\left| \frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) - 1 \right| = O\left(b_n \ln b_n / k_n\right).$$

Besides, under the condition $b_n \ln b_n = o(k_n)$, we have

$$\kappa_n^2(x) \sim k_n(b_n + 1), \ x \in [0,1].$$

For a proof, see for instance JACOB & SUQUET [23], equation (4.14).

5.2.4 Faber-Shauder estimate

Similarly to the Haar case, we suppose that the sequences (h_n) and (k_n) verify (5.6) where d_n is even. The kernel K_n^{F} can be explicitely computed (see Proposition 1 in GARDES [8]) although its expression is too heavy to be given here. The following results

$$\max_{1\leq r\leq k_n} |\kappa_{n,r}(x)| = O\left(b_n\right),$$

$$\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x) = 1,$$

$$\frac{1}{2} \leq \frac{\kappa_n^2(x)}{k_n(b_n+1)} \leq 2,$$

$(x \in [0,1])$ can be found in GARDES [8], Lemma 2.

5.3 Convergence results

We first study the convergence of the centered estimates. Proposition 1 can be rewritten as:

Corollary 2 *Suppose f is C^2. Under assumptions of Table 1, the Haar, Faber-Shauder, trigonometric and kernel estimates verify:*

i) for all $(x_1,...,x_d) \subset [0,1]$,

$$\left\{\frac{nc}{\kappa_n(x_j)}(\hat f_n(x_j) - E(\hat f_n(x_j)) : 1 \leq j \leq d\right\} \xrightarrow[d]{} N\left(0, I\right),$$

with $\kappa_n(x_j) \asymp (b_n k_n)^{1/2}$, $j = 1,\ldots,d$.

ii) for all $(x_1,...,x_d) \subset [0,1]$, and all $\varepsilon_n \downarrow 0$ such that $b_n/k_n = o\left(\varepsilon_n\right)$,

$$\left\{\sqrt{\varepsilon_n}\frac{nc}{\kappa_n\left(x_j\right)}\left(\hat f_n\left(x_j\right) - E\left(\hat f_n\left(x_j\right)\right)\right) : 1 \leq j \leq d\right\}.$$

follows the large deviation principle in $\mathbb{R}^d$ with speed (ε_n) and good rate function $I(u) = \|u\|_{\mathbb{R}^d}^2/2$.

The notation $\kappa_n(x_j) \asymp (b_n k_n)^{1/2}$ means that

$$0 < \liminf \kappa_n(x_j)/(b_n k_n)^{1/2} \leq \limsup \kappa_n(x_j)/(b_n k_n)^{1/2} < \infty.$$

In the case of the Haar, kernel and trigonometric estimate, this reduces to $\kappa_n(x_j) \sim (b_n k_n)^{1/2}$. The first line of assumptions of Table 1 ensures that the mean number of points in each cell $D_{n,r}$ converges to infinity. The second line of assumptions determines the minimal amount of smoothing required on each interval $I_{n,r}$ for asymptotic normality. Finally, the third line of assumptions imposes that the mean number of points above $m_{n,r}$ in each cell $D_{n,r}$ converges to 0, so that the function f could be approximated by a constant on $I_{n,r}$.

Proof : Assumptions **(H.1)**, **(H.2)** and **(H.4)** are easily verified thanks to the results of the previous paragraph. Let us focus on assumption **(H.3)** and note

$$\sigma_{i,j}^{(n)} = \sum_{r=1}^{k_n} w_{n,r}(x_i) w_{n,r}(x_j),\ 1 \leq i,j \leq d.$$

It is clear that one always have $\sigma_{i,i}^{(n)} = 1$, $1 \leq i \leq d$. The computation of $\sigma_{i,j} = \lim_{n\to\infty} \sigma_{i,j}^{(n)}$, $i \neq j$ is done separately for each estimate.

- Haar estimate: Let $\ell(x_i)$ and $\ell(x_j)$ such that $x_i \in J_{\ell(x_i)}$ and $x_j \in J_{\ell(x_j)}$. Then, in view of (5.7),

$$K_n^{\mathrm{H}}(x_i, x_{n,r}) K_n^{\mathrm{H}}(x_j, x_{n,r}) = (b_n+1)^2 \mathbb{I}_{\{x_{n,r} \in J_{\ell(x_i)}\}} \mathbb{I}_{\{x_{n,r} \in J_{\ell(x_j)}\}}.$$

  Now, for n large enough, $J_{\ell(x_i)} \cap J_{\ell(x_j)} = \emptyset$ and then $\sigma_{i,j}^{(n)} = 0$. Thus $\sigma_{i,j} = 1$ if $i = j$, and $\sigma_{i,j} = 0$ otherwise. The case of the Faber-Shauder estimate is similar.

- Trigonometric estimate: Equation (4.35) of Jacob & Suquet [23] entails $\sigma_{i,j} = 0$ for $i \neq j$ when $b_n \ln b_n = o(k_n)$.

- Kernel estimate: Girard & Jacob [14], Corollary 1 shows that $\sigma_{i,j} = 0$ for $i \neq j$ when $b_n^{3/2} = o(k_n)$. ■

The convergence of the estimates centered on the true function f is now derived from Theorem 1 and Theorem 2 at the expense on additionary conditions on (b_n) and (k_n).

Corollary 3 *Suppose f is C^2. Under assumptions of Table 2, the Haar, Faber-Shauder, trigonometric and kernel estimates verify:*
i) for all $(x_1, ..., x_d) \subset [0,1]$,

$$\left\{ \frac{nc}{\kappa_n(x_j)} (\hat{f}_n(x_j) - f(x_j)) : 1 \leq j \leq d \right\} \underset{d}{\to} N(0, I),$$

with $\kappa_n(x_j) \asymp (b_n k_n)^{1/2}$, $j = 1, \ldots, d$.
ii) for all $(x_1, ..., x_d) \subset [0,1]$, and all $\varepsilon_n \downarrow 0$ such that $b_n/k_n = o(\varepsilon_n)$,

$$\left\{ \sqrt{\varepsilon_n} \frac{nc}{\kappa_n(x_j)} \left(\hat{f}_n(x_j) - f(x_j) \right) : 1 \leq j \leq d \right\}.$$

follows the large deviation principle in $\mathbb{R}^d$ with speed (ε_n) and good rate function $I(u) = \|u\|_{\mathbb{R}^d}^2/2$.

Proof : Conditions **(H.5)** and **(H.7)** are straightfowardly verified thanks to the results of paragraph 5.2. Let us focus on condition **(H.6)** and consider the different estimates separately.

- Haar estimate: Let $\ell(x)$ such that $x \in J_{\ell(x)}$. In view of (5.7),

$$\begin{aligned}
\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(f(x)-m_{n,r}) &= \frac{1}{d_n}\sum_{r=1}^{k_n}(f(x)-m_{n,r})\mathbb{I}_{\{x_{n,r}\in J_{\ell(x)}\}} \\
&\leq \max_r \sup_{y\in J_{\ell(x)}} |f(y)-m_{n,r}| \\
&= O\left(1/b_n\right),
\end{aligned}$$

  and (5.8) concludes the proof. The case of the Faber-Shauder basis is similar.

- Trigonometric estimate: Consider the expansion

$$\begin{aligned}
\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(f(x)-m_{n,r}) &= \left(f(x)-\sum_{r=1}^{k_n}\kappa_{n,r}(x)\lambda_{n,r}\right) \\
&+ f(x)\left(\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)-1\right) \\
&+ \frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(k_n\lambda_{n,r}-m_{n,r}).
\end{aligned}$$

  The first term is bounded by GIRARD & JACOB [13], Proposition 2:

$$\left|f(x)-\sum_{r=1}^{k_n}\kappa_{n,r}(x)\lambda_{n,r}\right| = O\left(b_n \ln b_n/k_n\right)+O\left(1/b_n\right).$$

  The second term is controlled with **(H.5)**, and the third one by the Cauchy-Schwarz inequality

$$\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(k_n\lambda_{n,r}-m_{n,r}) = O\left(\kappa_n(x)/k_n^{3/2}\right).$$

- Kernel estimate: In a similar way,

$$\begin{aligned}
\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(f(x)-m_{n,r}) &= \left(f(x)-\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)f(x_{n,r})\right) \\
&+ f(x)\left(\frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)-1\right) \\
&+ \frac{1}{k_n}\sum_{r=1}^{k_n}\kappa_{n,r}(x)(f(x_{n,r})-m_{n,r}).
\end{aligned}$$

The first term is bounded by GIRARD & JACOB [14], Proposition 1:

$$\left| f(x) - \sum_{r=1}^{k_n} \kappa_{n,r}(x) f(x_{n,r}) \right| = O\left(b_n^3/k_n^2\right) + O\left(1/b_n^2\right).$$

The second term is controlled with **(H.5)**, and the third one by

$$\frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x)(f(x_{n,r}) - m_{n,r}) = \left(\frac{1}{k_n} \sum_{r=1}^{k_n} \kappa_{n,r}(x) \right) O\left(1/k_n\right) = O\left(1/k_n\right),$$

and (5.5) concludes the proof. ■

We conclude this investigation of the asymptotic properties of the estimates by providing the optimal choices of the (b_n) and (k_n) sequences in Table 3 and the resulting convergence rate of the estimate. From the asymptotic point of view, the kernel estimate is the better one, and the trigonometric estimate is the worst. The Haar and Faber-Shauder estimates share the same intermediary convergence rate. However, the convergence rate should not be the only criterion for choosing the estimator. For instance, prior information on the regularity of f can be a good indication for the choice of the estimator. Haar and Faber-Shauder estimates are well-adapted for the estimation of non smooth boundaries whereas, the trigonometric and kernel estimates are designed for regular boundaries. Besides, the trigonometric estimate is a good candidate for the estimation of periodic functions f. This kind of problem arises when a star shaped support has been reduced to the kind of domain (1.1) studied here by the transformation from polar to cartesian coordinates (see JACOB & ABBAR [22]). Let us also recall that the kernel estimate suffers from a boundary effects leading to poor estimations of f at the boundaries of the interval. Nevertheless, simple and efficient corrections exist, see e.g. GIRARD & JACOB [12]. We now illustrate the behaviour of the estimates on simulations.

5.4 Numerical experiments

In this illustration, the sum of $n = 200$ independent Poisson processes with the same intensity rate $c = 4$ is simulated on the set S defined by (1.1) with

$$f(x) = [0.1 + \sin(\pi x)] \left[1.1 - 0.5 \exp\left(-64(x - 0.5)^2\right)\right],$$

for $x \in [0, 1]$. The unit interval is divided into $k_n = 32$ equidistant intervals. In the case of the kernel estimate, we choose $h_n = 0.025$ for the smoothing parameter (leading to $b_n = 40$). In the case of the Haar, Faber-Shauder and trigonometric estimates, we consider an expansion of order $b_n = 15$ of f in the corresponding basis. The results are presented in Figure 1 for comparison. In each case, the estimate $\hat{f}_n$ is superimposed to the true function f and the simulated point process. We also take profit of Theorem 1 to draw 90% confidence intervals for $f(x)$ at 50 different locations. The qualitative differences between the four estimates appear clearly. The Haar estimate is piecewise constant (Figure 1(a)) and the Faber-Shauder estimate is piecewise linear (Figure 1(b)). Also note on Figure 1(c) the oscillations of the trigonometric estimate due to the periodic nature of the associated Dirichlet's kernel. It is apparent that the first peak is under-estimated by all the estimates. The smoothing induced by the different kernels is not sufficient to overcome the missing of simulated points in the peak. Nevertheless, the unknown function f is always included in the confidence intervals.

Haar & Faber-Shauder	Trigonometric	Kernel
$k_n = o(n)$	$k_n = o(n)$	$k_n = o(n)$
$b_n = o(k_n)$	$b_n \ln b_n = o(k_n)$	$b_n^{3/2} = o(k_n)$
$n = o\left(k_n^2\right)$	$n = o\left(k_n^2\right)$	$n = o\left(k_n^2\right)$

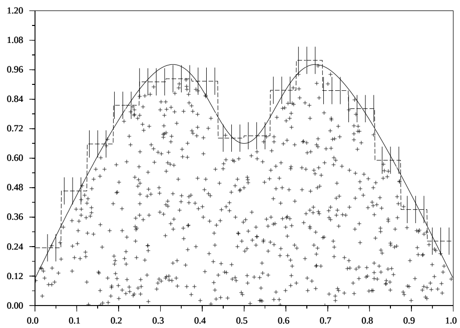
Table 1: Sufficient conditions **(H.1)**–**(H.4)** for the different estimators.

Haar & Faber-Shauder	Trigonometric	Kernel
$k_n = o(n/\ln n)$	$k_n = o(n/\ln n)$	$k_n = o(n/\ln n)$
$b_n = o(k_n)$	$b_n \ln b_n = o(k_n)$	$b_n^{3/2} = o(k_n)$ $k_n = o\left(b_n^{5/2}\right)$
$n = o\left(k_n^{1/2} b_n^{3/2}\right)$	$n = o\left(k_n^{1/2} b_n^{3/2}\right)$ $n = o\left(k_n^{3/2}/b_n^{1/2} \ln b_n\right)$	$n = o\left(k_n^{5/2}/b_n^{5/2}\right)$

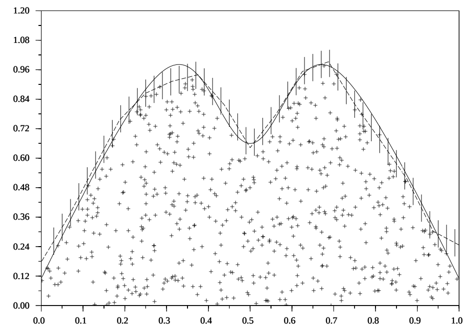
Table 2: Sufficient conditions **(H.1)**–**(H.7)** for the different estimators.

Estimate	Haar & Faber-Shauder	Trigonometric	Kernel
k_n	$n^{1/2}\rho_n^2$	$n^{4/5}(\ln n)^{2/3}\rho_n^2$	$n^{2/3}\rho_n^2$
b_n	$n^{1/2}$	$n^{2/5}$	$n^{4/15}$
κ_n/n	$n^{-1/2}\rho_n$	$n^{-2/5}(\ln n)^{1/3}\rho_n$	$n^{-8/15}\rho_n$

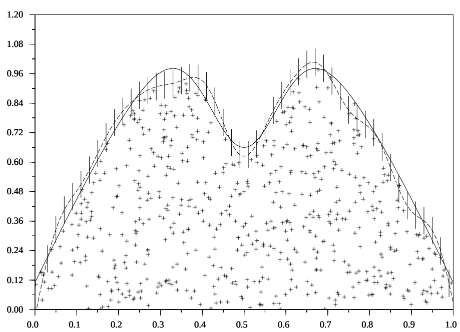
Table 3: Optimal choices of (b_n) and (k_n) sequences, and convergence rate of the estimators. (ρ_n) is an arbitrary positive sequence such that $\rho_n \to \infty$ and $\rho_n = o(\ln n)$.



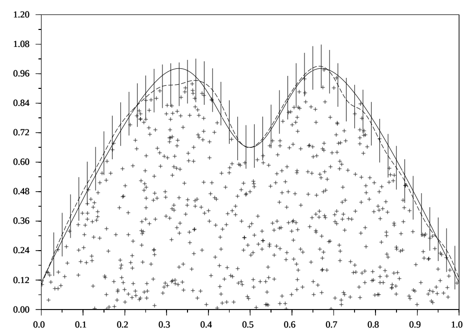
(a)



(b)



(c)



(d)

Figure 1: Superimposition of the simulated point process, the function f to estimate (continuous line) and the four estimates (dashed lines) obtained with the Haar (a), Faber-Shauder (b), trigonometric (c) and kernel (d) methods. In each case, the vertical lines represent the 90% confidence intervals.

6 Appendix

6.1 Multivariate central limit theorem and moderate deviations

In this part, we provide general theorems about the central limit property and the moderate deviation principle of a sequence of random $\mathbb{R}^d$ valued vectors

$$\theta_n = \sum_{r=1}^{k_n} w_{n,r} \zeta_{n,r}, \qquad n \geq 1,$$

where $(w_{n,r})_{1\leq r\leq k_n} \subset \mathbb{R}^d$ and $(\zeta_{n,r})_{1\leq r\leq k_n}$ are random variables such that:

(A.1) $(\zeta_{n,r})_{1\leq r\leq k_n}$ are centered and independent random variables.

(A.2) $\max\limits_{1\leq r\leq k_n} \left|\mathbb{E}\left(\zeta_{n,r}^2\right) - 1\right| \to 0.$

(A.3) There exists a covariance matrix Σ in $\mathbb{R}^d$ such that for all $\lambda \in \mathbb{R}^d$,

$$\sum_{r=1}^{k_n} \langle w_{n,r}, \lambda\rangle_{\mathbb{R}^d}^2 \to {}^t\lambda\Sigma\lambda.$$

(A.4) $\max\limits_{1\leq r\leq k_n} \|w_{n,r}\|_{\mathbb{R}^d} = o(1)$.

(A.5) $\limsup\limits_{\alpha\to\infty} \limsup\limits_{n\to\infty} \max\limits_{1\leq r\leq k_n} \mathbb{E}\left(\zeta_{n,r}^2 \mathbb{I}_{\{|\zeta_{n,r}|>\alpha\}}\right) = 0.$

(A.6) There exists $K > 0$ such that

$$\limsup_{n\to\infty} \max_{\ell\geq 3} \max_{1\leq r\leq k_n} \frac{1}{K^\ell \ell!} \mathbb{E}\left(|\zeta_{n,r}|^\ell\right) < 1.$$

Let us note that (A.6) implies (A.5).

Theorem 3 *Under assumptions* (A.1) – (A.5),

$$\theta_n \underset{d}{\to} N(0, \Sigma).$$

Proof : We have to show that, for all $\lambda \in \mathbb{R}^d$,

$$\langle\theta_n, \lambda\rangle_{\mathbb{R}^d} \underset{d}{\to} N\left(0,\ {}^t\lambda\Sigma\lambda\right). \tag{6.1}$$

Now, observe that (A.1)–(A.3) entail

$$\mathbb{V}\left(\langle\theta_n, \lambda\rangle_{\mathbb{R}^d}\right) = \sum_{r=1}^{k_n} \langle w_{n,r}, \lambda\rangle_{\mathbb{R}^d}^2 \mathbb{E}\left(\zeta_{n,r}^2\right) \ = \ {}^t\lambda\Sigma\lambda + o(1).$$

Hence, by Lindeberg Theorem (see e.g : DUDLEY [7], p. 248), it is easy to see that (6.1) holds whenever for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \sum_{r=1}^{k_n} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d}^2 \mathbb{E}\left(\zeta_{n,r}^2 \mathbb{I}_{\left\{ \left| \zeta_{n,r} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d} \right| > \varepsilon \right\}} \right) = 0.$$

Fix $\lambda \in \mathbb{R}^d$, $\varepsilon > 0$ and $\alpha > 0$. Using (A.4) , we get for all n large enough and all $1 \le r \le k_n$ that

$$\mathbb{I}_{\left\{ \left| \zeta_{n,r} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d} \right| > \varepsilon \right\}} \le \mathbb{I}_{\{ |\zeta_{n,r}| > \alpha \}}.$$

Hence,

$$\begin{aligned} &\limsup_{n\to\infty} \sum_{r=1}^{k_n} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d}^2 \mathbb{E}\left(\zeta_{n,r}^2 \mathbb{I}_{\left\{ \left| \zeta_{n,r} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d} \right| > \varepsilon \right\}} \right) \\ &\le \limsup_{n\to\infty} \left(\sum_{r=1}^{k_n} \langle w_{n,r}, \lambda \rangle_{\mathbb{R}^d}^2 \right) \max_{1 \le r \le k_n} \mathbb{E}\left(\zeta_{n,r}^2 \mathbb{I}_{\{ |\zeta_{n,r}| > \alpha \}} \right) \\ &\le {}^t \lambda \Sigma \lambda \limsup_{n\to\infty} \max_{1 \le r \le k_n} \mathbb{E}\left(\zeta_{n,r}^2 \mathbb{I}_{\{ |\zeta_{n,r}| > \alpha \}} \right). \end{aligned}$$

and we get the result by (A.5) when $\alpha \uparrow \infty$. ■

Let $\varepsilon_n \downarrow 0$, $\mathrm{I} : \mathbb{R}^d \to [0, +\infty]$ such that for all $\alpha > 0$, $\{\mathrm{I} \le \alpha\}$ is compact and recall that :

Definition 1 *A sequence of $\mathbb{R}^d$ valued random variable (Θ_n) is said to follow the large deviation principle in $\mathbb{R}^d$ with speed (ε_n) and good rate function I whenever for every set $A \in \mathcal{B}\left(\mathbb{R}^d\right)$,*

$$\begin{aligned} -\inf\left\{ \mathrm{I}(u) : u \in \overset{\circ}{A} \right\} &\le \liminf_{n\to\infty} \varepsilon_n \log \mathbb{P}(\Theta_n \in A) \\ &\le \limsup_{n\to\infty} \varepsilon_n \log \mathbb{P}(\Theta_n \in A) \\ &\le -\inf\left\{ \mathrm{I}(u) : u \in \overline{A} \right\}, \end{aligned}$$

where $\overset{\circ}{A}$ (resp. $\overline{A}$) denotes the interior (resp. closure) of A in $\mathbb{R}^d$.

Assume that Σ is regular and define

$$\mathrm{I}_\Sigma : u \in \mathbb{R}^d \mapsto \frac{1}{2}\, {}^t u \Sigma^{-1} u.$$

Theorem 4 *Assume that* (A.1) – (A.4) *and* (A.6) *hold, where the matrix* Σ *in* (A.3) *is regular, then, for all sequences* $\varepsilon_n \downarrow 0$ *such that*

$$\max_{1\leq r\leq k_n} \|w_{n,r}\|_{\mathbb{R}^d} = o\left(\sqrt{\varepsilon_n}\right), \tag{6.2}$$

$\left(\sqrt{\varepsilon_n}\theta_n\right)$ *follows the large deviation principle in* $\mathbb{R}^d$ *with speed* (ε_n) *and good rate function* I_Σ.

Proof : Denote by

$$\Lambda_n : \lambda \in \mathbb{R}^d \mapsto \varepsilon_n \log \mathbb{E}\left(\exp\left(\varepsilon_n^{-1}\left\langle \lambda, \sqrt{\varepsilon_n}\theta_n\right\rangle_{\mathbb{R}^d}\right)\right),$$

and set

$$\Lambda : \lambda \in \mathbb{R}^d \mapsto \frac{1}{2}\,{}^t\lambda\Sigma\lambda.$$

Then,

$$\Lambda_n(\lambda) = \varepsilon_n \sum_{r=1}^{k_n} \log\left(1 + \alpha_{n,r}(\lambda)\left(1 + \beta_{n,r}(\lambda)\right)\right),$$

with

$$\alpha_{n,r}(\lambda) = \frac{\langle \lambda, w_{n,r}\rangle_{\mathbb{R}^d}^2}{2\varepsilon_n}$$

and

$$\beta_{n,r}(\lambda) = \mathbb{E}\left(\zeta_{n,r}^2\right) - 1 + 2\mathbb{E}\left(\sum_{\ell=3}^{+\infty}\left(\frac{\langle \lambda, w_{n,r}\rangle_{\mathbb{R}^d}}{\sqrt{\varepsilon_n}}\right)^{\ell-2}\frac{\zeta_{n,r}^\ell}{\ell!}\right).$$

Now, by (6.2),

$$\max_{1\leq r\leq k_n} \alpha_{n,r}(\lambda) \to 0 \text{ as } n \uparrow \infty.$$

Moreover, by (A.2), (A.6) and (6.2), for all large n,

$$\begin{aligned}
\max_{1\leq r\leq k_n} \left|\beta_{n,r}(\lambda)\right| &\leq \max_{1\leq r\leq k_n}\left|\mathbb{E}\left(\zeta_{n,r}^2\right) - 1\right| + 2\max_{1\leq r\leq k_n}\sum_{\ell=3}^{+\infty}\left|\varepsilon_n^{-1/2}\langle \lambda, w_{n,r}\rangle_{\mathbb{R}^d}\right|^{\ell-2}\mathbb{E}\left(\left|\frac{\zeta_{n,r}^\ell}{\ell!}\right|\right) \\
&\leq o(1) + 2K^2\sum_{\ell=1}^{+\infty}\left(K\|\lambda\|_{\mathbb{R}^d}\varepsilon_n^{-1/2}\max_{1\leq r\leq k_n}\|w_{n,r}\|_{\mathbb{R}^d}\right)^\ell \\
&\leq o(1) + 4K^3\|\lambda\|_{\mathbb{R}^d}\varepsilon_n^{-1/2}\max_{1\leq r\leq k_n}\|w_{n,r}\|_{\mathbb{R}^d} = o(1).
\end{aligned}$$

Therefore, by Lemma 8 below and (A.3),

$$\begin{aligned}\Lambda_n(\lambda) &= \varepsilon_n \sum_{r=1}^{k_n} \alpha_{n,r}(\lambda) + o\left(\varepsilon_n \sum_{r=1}^{k_n} \alpha_{n,r}(\lambda)\right) = \frac{1}{2}\sum_{r=1}^{k_n} \langle \lambda, w_{n,r}\rangle_{\mathbb{R}^d}^2 + o\left(\sum_{r=1}^{k_n} \langle \lambda, w_{n,r}\rangle_{\mathbb{R}^d}^2\right) \\ &= \Lambda(\lambda) + o(1).\end{aligned}$$

Hence, by the Gartner-Ellis Theorem (see Theorem 2-3-6 of DEMBO & ZEITOUNI [4], p.44), $\left(\sqrt{\varepsilon_n}\theta_n\right)$ follows the large deviation principle in $\mathbb{R}^d$ with speed (ε_n) and good rate function

$$\Lambda^*(u) = \sup\left\{\langle \lambda, u\rangle_{\mathbb{R}^d} - \Lambda(\lambda) : \lambda \in \mathbb{R}^d\right\}$$

and it is easily seen that $\Lambda^* = \mathrm{I}_\Sigma$. ■

6.2 Technical lemmas

The following facts have been useful in our proofs :

Lemma 7 *For every $\ell \geq 1$ and every random variables $\chi_1, ..., \chi_p$ in $L^\ell(\mathbb{P})$,*

$$\mathbb{E}\left(|\chi_1 + ... + \chi_p|^\ell\right) \leq p^\ell \max_{1\leq j\leq p} \mathbb{E}\left(|\chi_j|^\ell\right).$$

In particular,

$$\mathbb{E}\left(|\chi_1 - \mathbb{E}(\chi_1)|^\ell\right) \leq 2^\ell \mathbb{E}\left(|\chi_1|^\ell\right).$$

The proof is a straightforward consequence of the $L^\ell(\mathbb{P})$ norm convexity.

Lemma 8 *Consider triangular arrays $(\alpha_{n,r})_{1\leq r\leq k_n} \subset \mathbb{R}^+$ and $(\beta_{n,r})_{1\leq r\leq k_n} \subset \mathbb{R}$ such that :*

$$\alpha_n := \max_{1\leq r\leq k_n} \alpha_{n,r} \to 0 \text{ as } n \uparrow \infty,$$

and

$$\beta_n := \max_{1\leq r\leq k_n} |\beta_{n,r}| \to 0 \text{ as } n \uparrow \infty.$$

Then,

$$\sum_{r=1}^{k_n} \log\left(1 + \alpha_{n,r}\left(1 + \beta_{n,r}\right)\right) = \sum_{r=1}^{k_n} \alpha_{n,r} + o\left(\sum_{r=1}^{k_n} \alpha_{n,r}\right).$$

Proof : Since for all $u \geq 0$,

$$0 \leq u - \log(1+u) \leq u^2,$$

we get, for all large n,

$$\begin{aligned}\left|\sum_{r=1}^{k_n} \log\left(1+\alpha_{n,r}\left(1+\beta_{n,r}\right)\right) - \sum_{r=1}^{k_n} \alpha_{n,r}\right| &\leq \sum_{r=1}^{k_n} \left[\alpha_{n,r}\left(1+\beta_{n,r}\right)\right]^2 + \left|\sum_{r=1}^{k_n} \alpha_{n,r}\beta_{n,r}\right| \\ &\leq (2\alpha_n + \beta_n) \sum_{r=1}^{k_n} \alpha_{n,r},\end{aligned}$$

which gives the result. ■

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